R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

The Pan African Mathematics Olympiads (PAMO) are prestigious event of the African Mathematics Union (AMU) organized each year in an African Country where the best pupils in Mathematics of the Secondary Education who are less than twenty (20) years old, are invited to compete. While emulating the African Youth, it contributes to integration and allows the AMU to detect new talents in Mathematics in order to secure a changeover of quality.

PAN AFRICAN MATHEMATICS OLYMPIAD

PROBLEMS

2009-2019

PAMO 2019 Day 1

4 April 2019

Duration: 4 h 30 min

1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers defined as follows:

   - $a_0 = 3$, $a_1 = 2$, and $a_2 = 12$; and
   - $2a_{n+3} - a_{n+2} - 8a_{n+1} + 4a_n = 0$ for $n \geq 0$.

   Show that a_n is always a strictly positive integer.

   (7 points)

2. Let k be a positive integer. Consider k not necessarily distinct prime numbers such that their product is ten times their sum. What are these primes and what is the value of k?

   (7 points)

3. Let ABC be a triangle, and D, E, F points on the segments BC, CA, AB respectively such that
$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}.$$
   Show that if the centres of the circumscribed circles of the triangles DEF and ABC coincide, then ABC is an equilateral triangle.

   (7 points)

La version française se trouve de l'autre côté de la page. *(Total: 21 points)*

PAMO 2019 Day 2

5 April 2019

Duration: 4 h 30 min

4. The tangents to the circumcircle of ΔABC at B and C meet at D. The circumcircle of ΔBCD meets sides AC and AB again at E and F respectively. Let O be the circumcentre of ΔABC. Show that AO is perpendicular to EF.

(7 points)

5. A square is divided into N^2 equal smaller non-overlapping squares, where $N \geq 3$. We are given a broken line which passes through the centres of all the smaller squares (such a broken line may intersect itself).

   (a) Show that it is possible to find a broken line composed of 4 segments for $N = 3$.

   (b) Find the minimum number of segments in this broken line for arbitrary N.

(7 points)

6. Find the 2019th strictly positive integer n such that $\binom{2n}{n}$ is not divisible by 5.

(7 points)

La version française se trouve de l'autre côté de la page. *(Total: 21 points)*

Kenya – PAMO 2018

26^{th} PAN AFRICAN MATHEMATICS OLYMPIAD

Nairobi from 23 to 30 June 2018

Day 1 : Wednesday, June 27, 2018 **Duration : 4 h 30 min**

Problem 1

Find all functions $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $(f(x+y))^2 = f(x^2) + f(y^2)$ for all $x, y \in \mathbb{Z}$.

Problem 2

A chess tournament is held with the participation of boys and girls. The girls are twice as many as boys. Each player plays against each other player exactly once. By the end of the tournament, there were no draws and the ratio of girl winnings to boy winnings was $\frac{7}{9}$.
How many players took part at the tournament ?

Problem 3

For any positive integer x, we set

$$g(x) = \text{the largest odd divisor of } x,$$
$$f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)} & \text{if } x \text{ is even;} \\ 2^{\frac{x+1}{2}} & \text{if } x \text{ is odd.} \end{cases}$$

Consider the sequence $(x_n)_{n \in \mathbb{N}}$ defined by $x_1 = 1,\ x_{n+1} = f(x_n)$. Show that the integer 2018 appears in this sequence, determine the least integer n such that $x_n = 2018$, and determine whether n is unique or not.

Kenya – PAMO 2018

26^{ièmes} **OLYMPIADES PAN AFRICAINES DE MATHÉMATIQUES**

Nairobi du 23 **au** 30 **Juin** 2018

Jour 1 : Mercredi 27 Juin 2018 **Durée : 4 h 30 min**

Problème 1

Trouver toutes les fonctions $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ telles que $(f(x+y))^2 = f(x^2) + f(y^2)$ pour tous $x, y \in \mathbb{Z}$.

Problème 2

Un tournoi d'échecs est organisé avec la participation de garçons et de filles. Le nombre de filles est le double de celui des garçons. Deux joueurs se rencontrent exactement une fois. A la fin du tournoi, il n'y a eu aucun nul et le rapport des victoires des filles par les victoires des garçons a été $\frac{7}{9}$.
Combien de joueurs ont participé au tournoi ?

Problème 3

Pour tout entier naturel non nul x, on pose

$$g(x) = \text{le plus grand diviseur impair de } x,$$
$$f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)}, & \text{si } x \text{ est pair;} \\ 2^{\frac{x+1}{2}}, & \text{si } x \text{ est impair.} \end{cases}$$

On considère la suite $(x_n)_{n \in \mathbb{N}^*}$ définie par $x_1 = 1,\ x_{n+1} = f(x_n)$. Montrer que l'entier 2018 apparaît dans cette suite, déterminer le plus petit entier naturel non nul n tel que $x_n = 2018$, et déterminer si n est unique ou non.

26^{th} PAN AFRICAN MATHEMATICS OLYMPIAD

Nairobi from 23 to 30 June 2018

Day 2 : Thursday, June 28, 2018 **Duration : 4 h 30 min**

Problem 4

Given a triangle ABC, let D be the intersection of the line through A perpendicular to AB, and the line through B perpendicular to BC. Let P be a point inside the triangle. Show that $DAPB$ is cyclic if and only if $\angle BAP = \angle CBP$.

Problem 5

Let a, b, c and d be non-zero pairwise different real numbers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4 \text{ and } ac = bd.$$

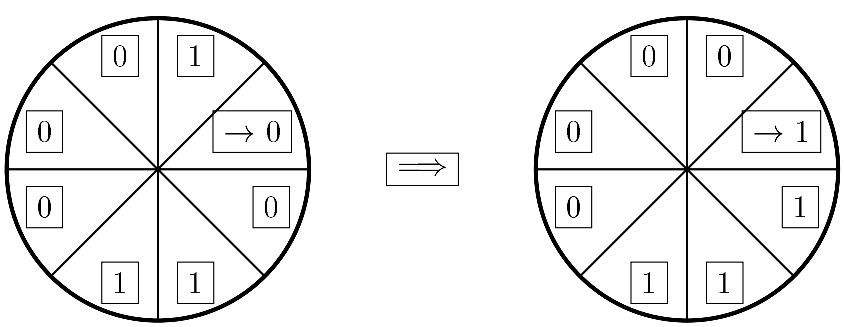
Show that

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} \leqslant -12$$

and that -12 is the maximum.

Problem 6

A circle is divided into n sectors ($n \geqslant 3$). Each sector can be filled in with either 1 or 0. Choose any sector $\mathcal{C}$ occupied by 0, change it into a 1 and simultaneously change the symbols $x,\ y$ in the two sectors adjacent to $\mathcal{C}$ to their complements $1-x,\ 1-y$. We repeat this process as long as there exists a zero in some sector. In the initial configuration there is a 0 in one sector and 1s elsewhere. For which values of n can we end this process ?



$26^{ièmes}$ **OLYMPIADES PAN AFRICAINES DE MATHÉMATIQUES**

Nairobi du 23 au 30 Juin 2018

<u>Jour 2 : Jeudi 28 Juin 2018</u> **Durée : 4 h 30 min**

Problème 4

Etant donné un triangle ABC, soit D le point d'intersection de la droite passant par A perpendiculaire à (AB), et de la droite passant par B perpendiculaire à (BC). Soit P un point à l'intérieur du triangle. Montrer que les points D, A, P et B sont cocycliques si et seulement si $\widehat{BAP} = \widehat{CBP}$.

Problème 5

Soient a, b, c et d des réels non nuls, deux à deux distincts tels que

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4 \text{ et } ac = bd.$$

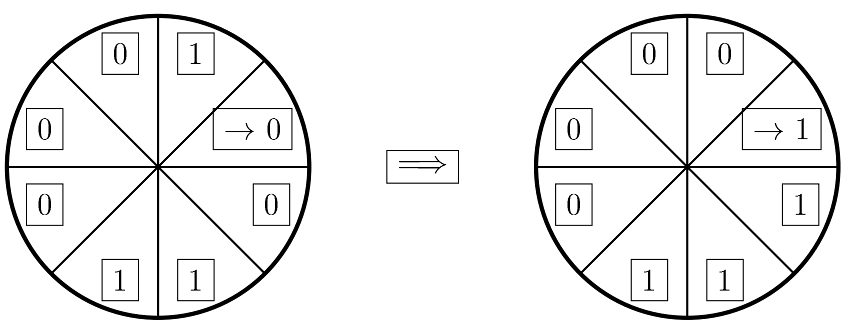
Montrer que

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} \leqslant -12$$

et que -12 est le maximum.

Problème 6

Un cercle est divisé en n secteurs ($n \geqslant 3$). Chaque secteur peut être rempli soit par 1 ou 0. On choisit n'importe quel secteur $\mathcal{C}$ contenant 0, on le change en 1 et on change simultanément les symboles x, y dans les deux secteurs adjacents à $\mathcal{C}$ en leurs complémentaires $1-x$, $1-y$. On répète ce procédé tant qu'il existe un zéro dans un certain secteur. Dans la configuration initiale il existe un 0 dans un seul secteur et des 1 dans les autres secteurs. Pour quelles valeurs de n peut-on finir ce procédé ?



U.M.A

Commission OPAM

25^{th} PAN AFRICAN MATHEMATICS OLYMPIAD

Rabat from 1 to 7 july 2017

Day 1 : Tuesday, july 4, 2017 **Duration : 4 h 30 min**

Problem 1

We consider the real sequence (x_n) defined by $x_0 = 0,\ x_1 = 1$ and $x_{n+2} = 3x_{n+1} - 2x_n$ for $n = 0, 1, \ldots$.
We define the sequence (y_n) by $y_n = x_n^2 + 2^{n+2}$ for every nonnegative integer n.
Prove that for every $n > 0$, y_n is the square of an odd integer.

Problem 2

Let $x,\ y$ and z be positive real numbers such that $xy + yz + zx = 3xyz$.
Prove that $x^2y + y^2z + z^2x \geqslant 2(x + y + z) - 3$.
In which case do we have equality ?

Problem 3

Let n be a positive integer. Find, in terms of n, the number of pairs (x, y) of positive integers that are solutions of the equation : $x^2 - y^2 = 10^2.30^{2n}$.
Prove further that this number is never a square.

U.M.A

Commission OPAM

25^{th} PAN AFRICAN MATHEMATICS OLYMPIAD

Rabat from 1 to 7 july 2017

Day 2 : Wednesday, july 5, 2017 **Duration : 4 h 30 min**

Problem 4

Find all the real numbers x such that $\frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3}$, where $[x]$ denotes the integer part of x and $\{x\} = x - [x]$.
For example, $[2.5] = 2$, $\{2.5\} = 0.5$ and $[-1.7] = -2$, $\{-1.7\} = 0.3$.

Problem 5

The numbers from 1 to 2017 are written on a board. Deka and Farid play the following game : each of them, on his turn, erases one of the numbers. Anyone who erases a multiple of $2, 3$ or 5 loses and the game is over. Is there a winning strategy for Deka ?

Problem 6

Let ABC be a triangle with H its orthocenter. The circle with diameter $[AC]$ cuts the circumcircle of the triangle ABH at K. Prove that the point of intersection of the lines CK and BH is the midpoint of the segment $[BH]$.

U.M.A

Commission OPAM

OPAM 2016
Dakar Sénégal

24^{th} PAN AFRICAN MATHEMATICS OLYMPIAD

Day 1 : Wednesday, April 27, 2016 **Duration : 4 h 30 min**

PROBLEM 1

Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect each other at two distinct points M and N. A common tangent line touches $\mathcal{C}_1$ at P and $\mathcal{C}_2$ at Q, the line being closer to N than to M. The line PN meets the circle $\mathcal{C}_2$ again at the point R.
Prove that the line MQ is a bisector of the angle $\angle PMR$.

PROBLEM 2

We have a pile of 2016 cards and a hat. We take out one card, put it in the hat and then divide the remaining cards into two arbitrary non empty piles. In the next step, we choose one of the two piles, we move one card from this pile to the hat and then divide this pile into two arbitrary non empty piles.
This procedure is repeated several times : in the k-th step ($k > 1$) we move one card from one of the piles existing after the step $(k-1)$ to the hat and then divide this pile into two non empty piles.
Is it possible that after some number of steps we get all piles containing three cards each ?

PROBLEM 3

For any positive integer n, we define the integer $P(n)$ by :

$$P(n) = n(n+1)(2n+1)(3n+1)...(16n+1).$$

Find the greatest common divisor of the integers $P(1),\ P(2),\ P(3), \ldots, P(2016)$.

U.M.A

Commission OPAM

OPAM 2016
Dakar Sénégal

24^{th} PAN AFRICAN MATHEMATICS OLYMPIAD

Day 2 : Thursday, April 28, 2016 **Duration : 4 h 30 min**

Problem 1

Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{1}{(x+1)^2+y^2+1}+\frac{1}{(y+1)^2+z^2+1}+\frac{1}{(z+1)^2+x^2+1}\leq\frac{1}{2}.$$

Problem 2

Let $ABCD$ be a trapezium such that the sides AB and CD are parallel and the side AB is longer than the side CD. Let M and N be on the segments AB and BC respectively, such that each of the segments CM and AN divides the trapezium in two parts of equal area.

Prove that the segment MN intersects the segment BD at its midpoint.

Problem 3

Consider an $n \times n$ grid formed by n^2 unit squares. We define the center of a unit square as the intersection of its diagonals.

Find the smallest integer m such that, choosing any m unit squares in the grid, we always get four unit squares among them whose centers are vertices of a parallelogram.

23$^{\text{rd}}$ edition of the Pan African Mathematics Olympiad

Abuja: 20 August - 29 August, 2015

First Day: 24 August 2015

Duration : 4 h 30

---

1. Prove that

$$\sqrt{x-1}+\sqrt{2x+9}+\sqrt{19-3x}<9$$

for all real x for which the left-hand side is well defined.

---

2. A convex hexagon $ABCDEF$ is such that

$$AB=BC \quad CD=DE \quad EF=FA$$

and

$$\angle ABC=2\angle AEC \quad \angle CDE=2\angle CAE \quad \angle EFA=2\angle ACE.$$

Show that AD, CF and EB are concurrent.

---

3. Let $a_1, a_2, ..., a_{11}$ be integers. Prove that there are numbers $b_1, b_2, ..., b_{11}$, each b_i equal $-1, 0$ or 1, but not all being 0, such that the number

$$N=a_1b_1+a_2b_2+\cdots+a_{11}b_{11}$$

is divisible by 2015.

23rd edition of the Pan African Mathematics Olympiad

Abuja: 20 August - 29 August, 2015

Second Day: 25 August 2015

Duration : 4 h 30

---

4. For a positive integer n denote $d(n)$ its greatest odd divisor. Find the value of the sum $d(1008) + d(1009) + \cdots + d(2015)$.

---

5. There are seven cards in a hat, and on the card k there is a number 2^{k-1}, $k = 1, 2, ..., 7$. Solarin picks the cards up at random from the hat, one card at a time, until the sum of the numbers on cards in his hand exceeds 124. What is the most probable sum he can get?

---

6. Let $ABCD$ be a quadrilateral (with non-perpendicular diagonals).

- **The perpendicular from A to BC meets CD at K.**
- **The perpendicular from A to CD meets BC at L.**
- **The perpendicular from C to AB meets AD at M.**
- **The perpendicular from C to AD meets AB at N.**

1. **Prove that KL is parallel to MN.**
2. **Prove that $KLMN$ is a parallelogram if $ABCD$ is cyclic.**

UNION MATHEMATIQUE AFRICAINE
Commission Olympiades Pan Africaines de Mathématiques

22nd edition of the Pan African Mathematics Olympiad

Abuja: 23 June – 2 July, 2013

First Day: 28 June 2013

Duration : 4 h 30

Exercise 1

A positive integer n is such that $n(n+2013)$ is a perfect square.

a) Show that n cannot be prime.

b) Find a value of n such that $n(n+2013)$ is a perfect square.

Exercise 2

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)f(y)+f(x+y)=xy$ for all real numbers x and y.

Exercise 3

Let $ABCDEF$ be a convex hexagon with $\angle A=\angle D$ and $\angle B=\angle E$. Let K and L be the midpoints of the sides AB and DE respectively.

Prove that the sum of the areas of triangles FAK, KCB and CFL is equal to half of the area of the hexagon if and only if

$$\frac{BC}{CD}=\frac{EF}{FA}$$

UNION MATHEMATIQUE AFRICAINE
Commission Olympiades Pan Africaines de Mathématiques

22nd edition of the Pan African Mathematics Olympiad

Abuja: 23 June – 2 July, 2013

Second Day: 29 June 2013

Duration : 4 h 30

Exercise 4

Let $ABCD$ be a convex quadrilateral with AB parallel to CD. Let P and Q be the midpoints of AC and BD, respectively.

Prove that if $\angle ABP = \angle CBD$, then $\angle BCQ = \angle ACD$.

Exercise 5

The cells of an $n \times n$ board with $n \geq 5$ are coloured black or white so that no three adjacent squares in a row, column or diagonal are the same colour. Show that for any 3×3 square within the board, two of its corner squares are coloured black and two are coloured white.

Exercise 6

Let x, y, and z be real numbers such that $x < y < z < 6$. Solve the system of inequalities:

$$\begin{cases} \dfrac{1}{y-x} + \dfrac{1}{z-y} \leq 2 \\ \dfrac{1}{6-z} + 2 \leq x \end{cases}$$

UNION MATHEMATIQUE AFRICAINE
Commission Olympiades Pan Africaines de Mathématiques

ATSM
ASSOCIATION TUNISIENNE DES SCIENCES MATHEMATIQUES

21st edition of Panafrican Mathematics Olympiad

Tunisia: 8 – 16 September, 2012

First Day: 12th September, 2012

Duration : 4 h 30

Exercise 1

AB is a chord (not a diameter) of a circle with centre *O*. Let *T* be a point on segment *OB*. The line through *T* perpendicular to *OB* meets *AB* at *C* and the circle at *D* and *E*. Denote by *S* the orthogonal projection of *T* onto *AB*.

Prove that $AS \cdot BC = TE \cdot TD$.

Exercise 2

Find all positive integers *m* and *n* such that $n^m - m$ divides m^2+2m.

Exercise 3

Find all real solutions *x* to the equation $[x^2 - 2x] + 2[x] = [x]^2$.

(Here $[a]$ denotes the largest integer less than or equal to *a*. For example $[7] = 7$, $[7.3] = 7$ and $[-4.2] = -5$.)

UNION MATHEMATIQUE AFRICAINE
Commission Olympiades Pan Africaines de Mathématiques

ATSM
ASSOCIATION TUNISIENNE DES SCIENCES MATHEMATIQUES

21st edition of Panafrican Mathematics Olympiad

Tunisia: 8 – 16 September, 2012

Second Day: 13th September, 2012

Duration : 4 h 30

Exercise 4

The numbers $\frac{1}{1}, \frac{1}{2}, \dots, \frac{1}{2012}$ are written on the blackboard. Aïcha chooses any two numbers from the blackboard, say x and y, erases them and she writes instead the number $x + y + xy$. She continues to do this until only one number is left on the board.
What are the possible values of the final number?

Exercise 5

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\boldsymbol{f(x^2 - y^2) = (x + y)(f(x) - f(y))}$ for all real numbers x and y.

Exercise 6

(i) Find the angles of ΔABC if the length of the altitude through B is equal to the length of the median through C and the length of the altitude through C is equal to the length of the median through B.

(ii) Find all possible values of $\angle ABC$ of ΔABC if the length of the altitude through A is equal to the length of the median through C and the length of the altitude through C is equal to the length of the median through B.

AFRICAN MATHEMATICAL UNION
Commission for Pan African Mathematics Olympiads

REPUBLIQUE DE CÔTE D'IVOIRE
Union – Discipline - Travail

MINISTERE DE L'EDUCATION NATIONALE

20th Pan African Mathematics Olympiads
Yamoussoukro, 20 – 30 May 2010

First Day : 26th May 2010

Duration : 4 h 30

Instructions

- *No calculating devices (computers, calculators, slide rules, etc), books, notes (printed or handwritten) are allowed in the examination room.*
- *Pens, pencils, rulers and compasses* ***only*** *may be used.*

Exercise 1

a) Show that it is possible to pair off the numbers 1, 2, 3, … , 10 so that the sums of each of the five pairs are five different prime numbers.
b) Is it possible to pair off the numbers 1, 2, 3, … , 20 so that the sums of each of the ten pairs are ten different prime numbers?

Exercise 2

How many ways are there to line up 19 girls (all of different heights) in a row so that no girl has a shorter girl both in front of and behind her?

Exercise 3

In an acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that BM = CF and <MBC = <FCA, prove that triangle ABC is equilateral.

AFRICAN MATHEMATICAL UNION
Commission for Pan African Mathematics Olympiads

REPUBLIQUE DE CÔTE D'IVOIRE
Union – Discipline - Travail

MINISTERE DE L'EDUCATION NATIONALE

20th Pan African Mathematics Olympiads
Yamoussoukro, 20 – 30 May 2010

Second Day : 27th May 2010

Duration : 4 h 30

Instructions

- *No calculating devices (computers, calculators, slide rules, etc), books, notes (printed or handwritten) are allowed in the examination room.*
- *Pens, pencils, rulers and compasses **only** may be used.*

Exercise 4

Seven distinct points are marked on a circle of circumference c. Three of the points form an equilateral triangle and the other four form a square. Prove that at least one of the seven arcs into which the seven points divide the circle has length less than or equal to $\frac{c}{24}$.

Exercise 5

A sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of positive integers is constructed as follows:
• if the last digit of a_n is less than or equal to 5 then this digit is deleted and a_{n+1} is the number consisting of the remaining digits. (If a_{n+1} contains no digits the process stops.)
• otherwise $a_{n+1} = 9a_n$
Can one choose a_0 so that an infinite sequence is obtained?

Exercise 6

Does there exist a function $f : Z \to Z$ such that $f(x + f(y)) = f(x) - y$ for all integers x and y ?

Problem 1:

Do there exist numbers $x_1, x_2, \ldots, x_{2009}$ from the set $\{-1, 1\}$, such that

$$x_1x_2 + x_2x_3 + x_3x_4 + \ldots + x_{2008}x_{2009} + x_{2009}x_1 = 999\,?$$

Problem 2:

Point P lies inside a triangle ABC. Let D, E and F be reflections of the point P in the lines BC, CA and AB, respectively. Prove that if the triangle DEF is equilateral, then the lines AD, BE and CF intersect in a common point.

Problem 3:

Let x be a real number with the following property: for each positive integer q, there exists an integer p, such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{3q}\,.$$

Prove that x is an integer.

Problem 4:

Consider n children in a playground, where $n \geq 2$. Every child has a coloured hat, and every pair of children is joined by a coloured ribbon. For every child, the colour of each ribbon held is different, and also different from the colour of that child's hat. What is the minimum number of colours that needs to be used?

Problem 5:

Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ for which $f(0) = 0$ and

$$f(x^2 - y^2) = f(x)f(y) \qquad \text{for all } x, y \text{ with } x > y,$$

where $\mathbb{N}_0$ is the set $\{0, 1, 2, ...\}$.

Problem 6:

Points C, E, D and F lie on a circle with center O. Two chords CD and EF intersect at a point N. The tangents at C and D intersect at A, and the tangents at E and F intersect at B. Prove that $ON \perp AB$.